

AN ELEMENTARY PROOF OF A CONGRUENCE BY SKULA AND GRANVILLE

ROMEO MEŠTROVIĆ

ABSTRACT. Let $p \geq 5$ be a prime, and let $q_p(2) := (2^{p-1} - 1)/p$ be the Fermat quotient of p to base 2. The following curious congruence was conjectured by L. Skula and proved by A. Granville

$$q_p(2)^2 \equiv -\sum_{k=1}^{p-1} \frac{2^k}{k^2} \pmod{p}.$$

In this note we establish the above congruence by entirely elementary number theory arguments.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

The Fermat Little Theorem states that if p is a prime and a is an integer not divisible by p , then $a^{p-1} \equiv 1 \pmod{p}$. This gives rise to the definition of the *Fermat quotient* of p to base a

$$q_p(a) := \frac{a^{p-1} - 1}{p},$$

which is an integer. Fermat quotients played an important role in the study of cyclotomic fields and Fermat Last Theorem. More precisely, divisibility of Fermat quotient $q_p(a)$ by p has numerous applications which include the Fermat Last Theorem and squarefreeness testing (see [1], [2], [3], [7] and [9]). Ribenboim [10] and Granville [7], besides proving new results, provide a review of known facts and open problems.

By a classical Glaisher's result (see [4] or [5]) for a prime $p \geq 3$,

$$(1.1) \quad q_p(2) \equiv -\frac{1}{2} \sum_{k=1}^{p-1} \frac{2^k}{k} \pmod{p}.$$

Recently Skula conjectured that for any prime $p \geq 5$,

$$(1.2) \quad q_p(2)^2 \equiv -\sum_{k=1}^{p-1} \frac{2^k}{k^2} \pmod{p}.$$

Applying certain polynomial congruences, Granville [5] proved the congruence (1.2). In this note, we give an elementary proof of this congruence which is based on congruences for some harmonic type sums.

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Remark 1.1. Recently, given a prime p and a positive integer $r < p - 1$, R. Tauraso [14, Theorem 2.3] established the congruence $\sum_{k=1}^{p-1} 2^k/k^r \pmod{p}$ in terms of an alternating r -tuple harmonic sum. For example, combining this result when $r = 2$ with the congruence (1.2) [14, Corollary 2.4], it follows that

$$\sum_{1 \leq i < j \leq p-1} \frac{(-1)^j}{ij} \equiv q_p(2)^2 \equiv -\sum_{k=1}^{p-1} \frac{2^k}{k^2} \pmod{p}.$$

2. PROOF OF THE CONGRUENCE (1.2)

The *harmonic numbers* H_n are defined by

$$H_n := \sum_{j=1}^n \frac{1}{j}, \quad n = 1, 2, \dots,$$

where by convention $H_0 = 0$.

Lemma 2.1. *For any prime $p \geq 5$ we have*

$$(2.1) \quad q_p(2)^2 \equiv \sum_{k=1}^{p-1} \left(2^k + \frac{1}{2^k}\right) \frac{H_k}{k+1} \pmod{p}.$$

Proof. In the present proof we will always suppose that i and j are positive integers such that $i \leq p-1$ and $j \leq p-1$, and that all the summations including i and j range over the set of such pairs (i, j) .

Using the congruence (1.1) and the fact that by Fermat Little Theorem, $2^{p-1} \equiv 1 \pmod{p}$, we get

$$\begin{aligned} q_p(2)^2 &= \left(\frac{2^{p-1} - 1}{p}\right)^2 \equiv \frac{1}{4} \left(\sum_{k=1}^{p-1} \frac{2^k}{k}\right)^2 = \frac{1}{4} \left(\sum_{k=1}^{p-1} \frac{2^{p-k}}{p-k}\right)^2 \\ (2.2) \quad &\equiv \frac{1}{4} \left(2 \sum_{k=1}^{p-1} \frac{2^{(p-1)-k}}{-k}\right)^2 \equiv \left(\sum_{k=1}^{p-1} \frac{1}{k \cdot 2^k}\right)^2 \\ &= \sum_{i+j \leq p} \frac{1}{ij \cdot 2^{i+j}} + \sum_{i+j \geq p} \frac{1}{ij \cdot 2^{i+j}} - \sum_{i+j=p} \frac{1}{ij \cdot 2^{i+j}} \\ &:= S_1 + S_2 - S_3 \pmod{p}. \end{aligned}$$

We will determine S_1 , S_2 and S_3 modulo p as follows.

$$\begin{aligned} S_1 &= \sum_{i+j \leq p} \frac{1}{ij \cdot 2^{i+j}} = \sum_{k=2}^p \sum_{i+j=k} \frac{1}{ij \cdot 2^k} \\ (2.3) \quad &= \sum_{k=2}^p \frac{1}{2^k} \cdot \frac{1}{k} \sum_{i=1}^{k-1} \left(\frac{1}{i} + \frac{1}{k-i}\right) = \sum_{k=2}^p \frac{2H_{k-1}}{k \cdot 2^k} = \sum_{k=1}^{p-1} \frac{H_k}{(k+1)2^k}. \end{aligned}$$

Observe that the pair (i, j) satisfies $i + j = k$ for some $k \in \{p, p+1, \dots, 2p-2\}$ if and only if for such a k holds $(p-i) + (p-j) = l$ with $l := 2p-k \leq p$.

Accordingly, using the fact that by Fermat Little Theorem, $2^{2p} \equiv 2^2 \pmod{p}$, we have

$$\begin{aligned}
 S_2 &= \sum_{i+j \geq p} \frac{1}{ij \cdot 2^{i+j}} = \sum_{(p-i)+(p-j) \geq p}^{p-1} \frac{1}{(p-i)(p-j) \cdot 2^{(p-i)+(p-j)}} \\
 &\equiv \sum_{i+j \leq p} \frac{1}{ij \cdot 2^{2p-(i+j)}} \equiv \frac{1}{4} \sum_{i+j \leq p} \frac{2^{i+j}}{ij} = \frac{1}{4} \sum_{k=2}^p \sum_{i+j=k} \frac{2^k}{ij} \\
 (2.4) \quad &= \frac{1}{4} \sum_{k=2}^p \frac{2^k}{k} \sum_{i=1}^{k-1} \left(\frac{1}{i} + \frac{1}{k-i} \right) = \sum_{k=2}^p \frac{2^{k-1} H_{k-1}}{k} \\
 &= \sum_{k=1}^{p-1} \frac{2^k H_k}{k+1} \pmod{p}.
 \end{aligned}$$

By *Wolstenholme's theorem* (see, e.g., [15], [6]; for its generalizations see [11, Theorems 1 and 2]) if p is a prime greater than 3, then the numerator of the fraction $H_{p-1} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{p-1}$ is divisible by p^2 . Hence, we find that

$$\begin{aligned}
 S_3 &= \sum_{i+j=p} \frac{2^{i+j}}{ij} = 2^p \sum_{i=1}^{p-1} \frac{1}{i(p-i)} \\
 (2.5) \quad &= \frac{2^p}{p} \sum_{i=1}^{p-1} \left(\frac{1}{i} + \frac{1}{p-i} \right) = \frac{2^{p+1}}{p} H_{p-1} \equiv 0 \pmod{p}.
 \end{aligned}$$

Finally, substituting (2.3), (2.4) and (2.5) into (2.2), we immediately obtain (2.1). \square

Proof of the following result easily follows from the congruence $H_{p-1} \equiv 0 \pmod{p}$.

Lemma 2.2. [13, Lemma 2.1] *Let p be an odd prime. Then*

$$(2.6) \quad H_{p-k-1} \equiv H_k \pmod{p}$$

for every $k = 1, 2, \dots, p-2$.

Lemma 2.3. *For any prime $p \geq 5$ we have*

$$(2.7) \quad q_p(2)^2 \equiv \sum_{k=1}^{p-1} \frac{H_k}{k \cdot 2^k} - \sum_{k=1}^{p-1} \frac{2^k}{k^2} \pmod{p}.$$

Proof. Since by Wolstenholme's theorem, $H_{p-1}/p \equiv 0 \pmod{p}$, using this and the congruences $2^{p-1} \equiv 1 \pmod{p}$ and (2.6) of Lemma 2.2, we immediately obtain

$$\begin{aligned}
 \sum_{k=1}^{p-1} \frac{2^k H_k}{k+1} &\equiv \sum_{k=1}^{p-2} \frac{2^k H_k}{k+1} = \sum_{k=1}^{p-2} \frac{2^{p-k-1} H_{p-k-1}}{p-k} \\
 (2.8) \quad &\equiv - \sum_{k=1}^{p-2} \frac{H_k}{k \cdot 2^k} \equiv - \sum_{k=1}^{p-1} \frac{H_k}{k \cdot 2^k} \pmod{p}.
 \end{aligned}$$

Further, we have

$$\begin{aligned}
 \sum_{k=1}^{p-2} \frac{H_k}{(k+1)2^k} &= 2 \sum_{k=1}^{p-2} \frac{H_{k+1} - \frac{1}{k+1}}{(k+1)2^{k+1}} \\
 (2.9) \qquad \qquad \qquad &= 2 \sum_{k=1}^{p-1} \frac{H_k}{k \cdot 2^k} - 2 \sum_{k=1}^{p-1} \frac{1}{k^2 \cdot 2^k}.
 \end{aligned}$$

Moreover, from $2^p \equiv 2 \pmod{p}$ we have

$$\begin{aligned}
 \sum_{k=1}^{p-1} \frac{1}{k^2 \cdot 2^k} &= \sum_{k=1}^{p-1} \frac{1}{(p-k)^2 \cdot 2^{p-k}} \\
 (2.10) \qquad \qquad \qquad &\equiv \sum_{k=1}^{p-1} \frac{1}{k^2 \cdot 2^{1-k}} = \frac{1}{2} \sum_{k=1}^{p-1} \frac{2^k}{k^2} \pmod{p}.
 \end{aligned}$$

The congruences (2.8), (2.9) and (2.10) immediately yield

$$\begin{aligned}
 \sum_{k=1}^{p-1} \left(2^k + \frac{1}{2^k}\right) \frac{H_k}{k+1} &= \sum_{k=1}^{p-1} \frac{2^k H_k}{k+1} + \sum_{k=1}^{p-1} \frac{H_k}{(k+1)2^k} \\
 (2.11) \qquad \qquad \qquad &\equiv \sum_{k=1}^{p-1} \frac{H_k}{k \cdot 2^k} - \sum_{k=1}^{p-1} \frac{2^k}{k^2} \pmod{p}.
 \end{aligned}$$

Finally, comparing (2.1) of Lemma 2.1 with (2.11), we obtain the desired congruence (2.7). \square

Notice that the congruence $\sum_{k=1}^{p-1} \frac{H_k}{k \cdot 2^k} \equiv 0 \pmod{p}$ with a prime $p \geq 5$ is recently established by Z.W. Sun [13, Theorem 1.1 (1.1)] and it is based on the identity from [13, Lemma 2.4]. Here we give another simple proof of this congruence (Lemma 2.6).

Lemma 2.4. *For any prime $p \geq 5$ we have*

$$(2.12) \qquad \sum_{k=1}^{p-1} \frac{H_k}{k \cdot 2^k} \equiv \frac{1}{2} \sum_{1 \leq i \leq j \leq p-1} \frac{2^i - 1}{ij} \pmod{p}.$$

Proof. From the identity

$$\left(\sum_{k=1}^{p-1} \frac{1}{k} \right) \left(\sum_{k=1}^{p-1} \frac{1}{k \cdot 2^k} \right) = \sum_{1 \leq i < j \leq p-1} \frac{1}{ij \cdot 2^j} + \sum_{1 \leq j < i \leq p-1} \frac{1}{ij \cdot 2^j} + \sum_{k=1}^{p-1} \frac{1}{k^2 \cdot 2^k},$$

and the congruence $H_{p-1} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p-1} \equiv 0 \pmod{p}$ it follows that

$$(2.13) \qquad \sum_{1 \leq i < j \leq p-1} \frac{1}{ij \cdot 2^j} + \sum_{1 \leq j < i \leq p-1} \frac{1}{ij \cdot 2^j} + \sum_{k=1}^{p-1} \frac{1}{k^2 \cdot 2^k} \equiv 0 \pmod{p}.$$

Since $2^p \equiv 2 \pmod{p}$, we have

$$\sum_{1 \leq j < i \leq p-1} \frac{1}{ij \cdot 2^j} \equiv \sum_{1 \leq j < i \leq p-1} \frac{1}{2} \frac{2^{p-j}}{(p-i)(p-j)} \equiv \frac{1}{2} \sum_{1 \leq i < j \leq p-1} \frac{2^j}{ij} \pmod{p},$$

which substituting into (2.13) gives

$$(2.14) \quad \sum_{1 \leq i < j \leq p-1} \frac{1}{ij \cdot 2^j} + \sum_{k=1}^{p-1} \frac{1}{k^2 \cdot 2^k} \equiv -\frac{1}{2} \sum_{1 \leq i < j \leq p-1} \frac{2^j}{ij} \pmod{p}.$$

Further, if we observe that

$$\sum_{k=1}^{p-1} \frac{H_k}{k \cdot 2^k} = \sum_{k=1}^{p-1} \frac{H_{k-1} + \frac{1}{k}}{k \cdot 2^k} = \sum_{1 \leq i < j \leq p-1} \frac{1}{ij \cdot 2^j} + \sum_{k=1}^{p-1} \frac{1}{k^2 \cdot 2^k},$$

then substituting (2.14) into the previous identity, we obtain

$$(2.15) \quad \sum_{k=1}^{p-1} \frac{H_k}{k \cdot 2^k} \equiv -\frac{1}{2} \sum_{1 \leq i < j \leq p-1} \frac{2^j}{ij} \pmod{p}.$$

Since

$$0 \equiv \left(\sum_{k=1}^{p-1} \frac{1}{k} \right) \left(\sum_{k=1}^{p-1} \frac{2^k}{k} \right) = \sum_{1 \leq j \leq i \leq p-1} \frac{2^j}{ij} + \sum_{1 \leq i < j \leq p-1} \frac{2^j}{ij} \pmod{p},$$

comparing this with (2.15), we immediately obtain

$$(2.16) \quad \sum_{k=1}^{p-1} \frac{H_k}{k \cdot 2^k} \equiv \frac{1}{2} \sum_{1 \leq i \leq j \leq p-1} \frac{2^i}{ij} \pmod{p}.$$

From a well known fact that (see e.g., [9, p. 353])

$$(2.17) \quad \sum_{k=1}^{p-1} \frac{1}{k^2} \equiv 0 \pmod{p}$$

we find that

$$\sum_{1 \leq i \leq j \leq p-1} \frac{1}{ij} = \frac{1}{2} \left(\left(\sum_{k=1}^{p-1} \frac{1}{k} \right)^2 + \sum_{k=1}^{p-1} \frac{1}{k^2} \right) \equiv 0 \pmod{p}.$$

Finally, the above congruence and (2.16) immediately yield the desired congruence (2.12). \square

Lemma 2.5. *For any positive integer n holds*

$$(2.18) \quad \sum_{1 \leq i \leq j \leq n} \frac{2^i - 1}{ij} = \sum_{k=1}^n \frac{1}{k^2} \binom{n}{k}.$$

Proof. Using the well known identities $\sum_{i=k}^j \binom{i-1}{k-1} = \binom{j}{k}$ and $\frac{1}{j} \binom{j}{k} = \frac{1}{k} \binom{j-1}{k-1}$ with $k \leq j$, and the fact that $\binom{i}{k} = 0$ when $i < k$, we have

$$\begin{aligned}
\sum_{1 \leq i \leq j \leq n} \frac{2^i - 1}{ij} &= \sum_{1 \leq i \leq j \leq n} \frac{(1+1)^i - 1}{ij} = \sum_{1 \leq i \leq j \leq n} \frac{1}{j} \sum_{k=1}^i \frac{1}{i} \binom{i}{k} \\
&= \sum_{1 \leq i \leq j \leq n} \frac{1}{j} \sum_{k=1}^n \frac{1}{k} \binom{i-1}{k-1} = \sum_{k=1}^n \frac{1}{k} \sum_{1 \leq i \leq j \leq n} \frac{1}{j} \binom{i-1}{k-1} \\
&= \sum_{k=1}^n \frac{1}{k} \sum_{k \leq i \leq j \leq n} \frac{1}{j} \binom{i-1}{k-1} = \sum_{k=1}^n \frac{1}{k} \sum_{j=i}^n \frac{1}{j} \sum_{i=k}^j \binom{i-1}{k-1} \\
&= \sum_{k=1}^n \frac{1}{k} \sum_{j=i}^n \frac{1}{j} \binom{j}{k} = \sum_{k=1}^n \frac{1}{k} \sum_{j=k}^n \frac{1}{k} \binom{j-1}{k-1} \\
&= \sum_{k=1}^n \frac{1}{k^2} \sum_{j=k}^n \binom{j-1}{k-1} = \sum_{k=1}^n \frac{1}{k^2} \binom{n}{k},
\end{aligned}$$

as desired. \square

Lemma 2.6. [13, Theorem 1.1 (1.1)] *For any prime $p \geq 5$ holds*

$$(2.19) \quad \sum_{k=1}^{p-1} \frac{H_k}{k \cdot 2^k} \equiv 0 \pmod{p}.$$

Proof. Using the congruence (2.12) from Lemma 2.4 and the identity (2.18) with $n = p - 1$ in Lemma 2.5, we find that

$$(2.20) \quad \sum_{k=1}^{p-1} \frac{H_k}{k \cdot 2^k} \equiv \sum_{k=1}^{p-1} \frac{1}{k^2} \binom{p-1}{k} \pmod{p}.$$

It is well known (see e.g., [8]) that for $k = 1, 2, \dots, p-1$,

$$(2.21) \quad \binom{p-1}{k} \equiv (-1)^k \pmod{p}.$$

Then from (2.20), (2.21) and (2.17) we get

$$\begin{aligned}
\sum_{k=1}^{p-1} \frac{H_k}{k \cdot 2^k} &\equiv \sum_{k=1}^{p-1} \frac{(-1)^k}{k^2} = \sum_{k=1}^{p-1} \frac{1}{k^2} - 2 \sum_{\substack{1 \leq j \leq p-1 \\ 2|j}} \frac{1}{j^2} \\
&\equiv -2 \sum_{\substack{1 \leq j \leq p-1 \\ 2 \nmid j}} \frac{1}{j^2} = -\frac{1}{2} \sum_{k=1}^{(p-1)/2} \frac{1}{k^2} \pmod{p}.
\end{aligned}$$

Finally, the above congruence together with a well known fact that (see e.g., [12, Corollary 5.2 (a) with $k = 2$])

$$\sum_{k=1}^{(p-1)/2} \frac{1}{k^2} \equiv 0 \pmod{p}$$

yields

$$\sum_{k=1}^{p-1} \frac{H_k}{k \cdot 2^k} \equiv 0 \pmod{p}.$$

This concludes the proof. \square

Proof of the congruence (1.2). The congruence (1.2) immediately follows from (2.7) of Lemma 2.3 and (2.19) of Lemma 2.6. \square

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MARITIME FACULTY, UNIVERSITY OF MONTENEGRO, DOBROTA 36, 85330 KOTOR, MONTENEGRO

E-mail address: romeo@ac.me